2. Basic assumptions for stellar atmospheres

1. geometry, stationarity
2. conservation of momentum, mass
3. conservation of energy
4. Local Thermodynamic Equilibrium

1. Geometry

Stars as gaseous spheres → **spherical symmetry**

Exceptions: rapidly rotating stars

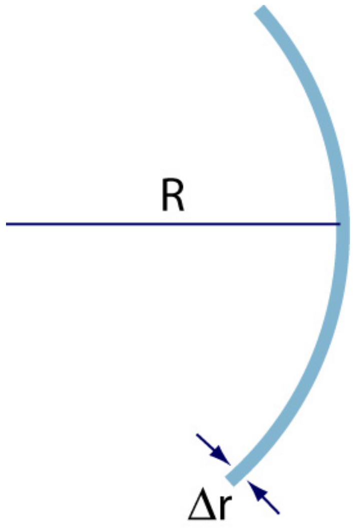
Be stars v_{rot} = 300 – 400 km/s

(Sun v_{rot} = 2 km/s)

For stellar photospheres typically: **$\Delta r/R \ll 1$**

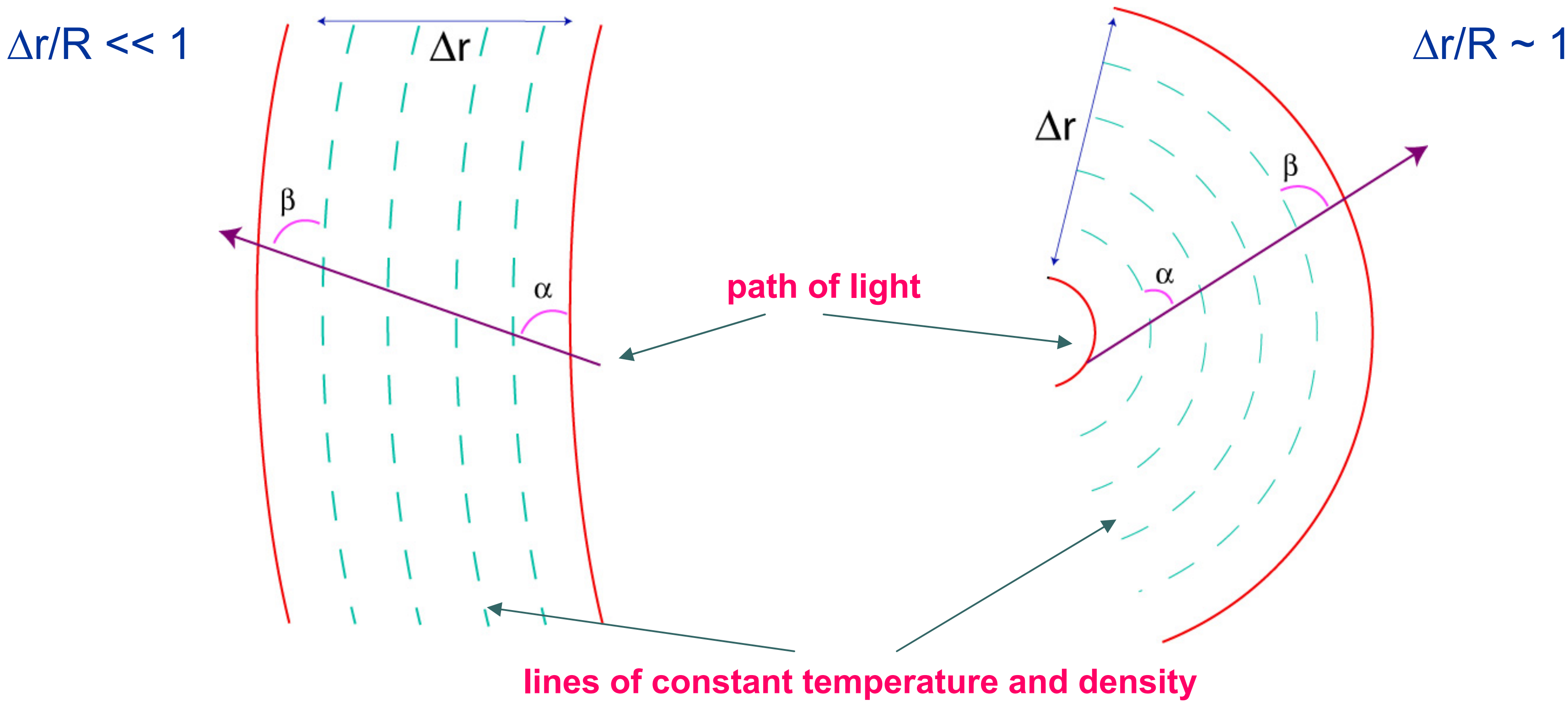
Sun: R_o = 700,000 km

photosphere	Δr = 300 km	$\Delta r/R = 4 \times 10^{-4}$
cromosphere	Δr = 3000 km	$\Delta r/R = 4 \times 10^{-3}$
corona		**$\Delta r/R \sim 3$**



As long as $\Delta r/R \ll 1$: plane-parallel symmetry

consider a light ray through the atmosphere:



Plane-parallel symmetry	Spherical symmetry
very small curvature ($\alpha \sim \beta$)	**significant curvature ($\alpha \neq \beta$)**
Solar photo/cromosphere, dwarfs, (giants)	**Solar corona, supergiants, expanding envelopes (OBA stars), novae, SN**

Homogeneity & stationarity

We assume the atmosphere to be **homogeneous**

Counter-examples:

sunspots, granulation, non-radial pulsations

clumps and shocks in hot star winds

magnetic Ap stars

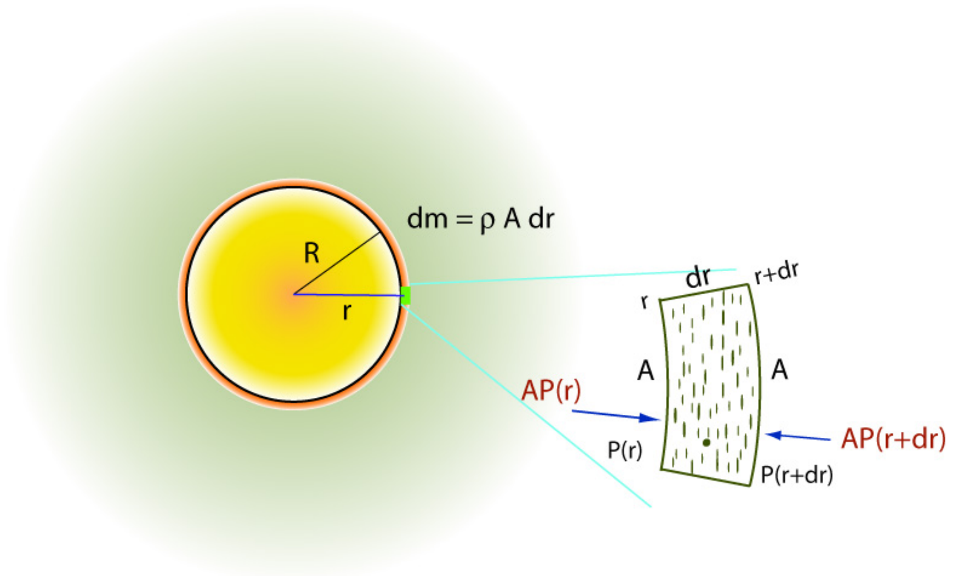
and **stationary**

Most spectra are time-independent: $\partial/\partial t = 0$ *(if we don't look carefully enough…)*

Exceptions: explosive phenomena (SN), stellar pulsations, magnetic stars, mass transfer in close binaries

2. Conservation of momentum & mass

consider a mass element dm in a spherically symmetric atmosphere



The acceleration of the mass element results from the sum of all forces acting on *dm*, according to Newton's 2nd law:

$$dm \frac{d}{dt} v(r,t) = \sum_i df_i = F$$

$$dm\frac{d}{dt}v(r,t) = \sum_i df_i = F$$

a. Hydrostatic equilibrium

Assuming a hydrostatic stratification: v(r,t)=0

$$0 = \sum_i df_i$$

Gravitational forces: $$df_{grav} = -G\frac{M_r \cdot dm}{r^2} = -g(r)\,dm$$

Gas Pressure forces: $$df_{p,gas} = -A\,P(r+dr) + A\,P(r) = -A\frac{dP}{dr}dr$$

Radiation forces: $$df_{rad} = g_{rad}(r)\,dm = \frac{\text{absorbed photon momentum}}{dt}$$

In equilibrium:

$$0 = \sum_i df_i = -g(r)dm - A\frac{dP}{dr}dr + g_{rad}dm$$

and substituting dm = A ρ dr:

$$-g(r)\rho A dr - A\frac{dP}{dr}dr + g_{rad}\rho A dr = 0$$

$$\frac{dP}{dr} = -\rho(r)[g(r) - g_{rad}]$$

Hydrostatic equilibrium in spherical symmetry

Approximation for g(r)

The mass within the atmosphere M(r) – M(R) << M(R)=M_*

$M(R) = M(r) = M_*$

$g(r) = G M_*/r^2$

Example: take a geometrically thin photosphere

$$\Delta M_{phot} \approx \bar{\rho}\frac{4\pi}{3}[(R+\Delta r)^3 - R^3] = \bar{\rho}\frac{4\pi}{3}[R^3(1+\frac{\Delta r}{R})^3 - R^3]$$

$$\approx \bar{\rho}\frac{4\pi}{3}R^3[1+3\frac{\Delta r}{R} - 1] = \bar{\rho}4\pi R^2 \Delta r$$

Example: the sun

$R = 7 \times 10^{10}$ cm, $\Delta r = 3 \times 10^{7}$ cm, $\rho \sim m_H N = 1.7 \times 10^{-24} \times 10^{15}$ g/cm^3:

$\Delta M_{phot} = 3 \times 10^{21}$ g $\qquad \Delta M_{phot}/M = 10^{-12}$

Moreover in plane-parallel symmetry:

$\Delta r/R << 1$

$g(r) = const$

$g = G M_*/R^2$

Main sequence star log g =	4 (cgs: cm/s^2)
supergiant	3.5 – 0.8
white dwarf	8
Sun	4.44
Earth	3.0

Barometric formula

Assume:

- plane-parallel symmetry
- $g_{rad} << g$
- ideal gas: $$P = \frac{\rho k T}{m_H \mu} = \frac{\rho R T}{\mu} = N_g k T$$
- and: $$\frac{1}{\rho}\frac{d\rho}{dr} >> \frac{1}{T}\frac{dT}{dr}$$

Note:

$$\frac{kT}{m_H \mu} = v_{sound}^2$$

k: Boltzmann constant = 1.38 x 10^{-16} erg K^{-1}

m_H: mass of H atom = 1.66 x 10^{-24} g

μ: mean molecular weight per free particle = <m> / m_H

R = k N_A = 8.314 x 10^7 erg/mole/K

⇒

$$\frac{dP}{dr} = -\rho(r) g(r)$$

Barometric formula

$$P = \frac{\rho k T}{m_H \mu} \qquad \frac{dP}{dr} = -\rho(r) g(r)$$

$$\frac{k}{m_H \mu}[T \frac{d\rho}{dr} + \rho \frac{dT}{dr}] = -g\rho$$

$$\frac{kT}{m_H \mu}[\frac{1}{\rho}\frac{d\rho}{dr} + \frac{1}{T}\frac{dT}{dr}] = -g$$

negligible by assumption

$$\frac{1}{\rho}\frac{d\rho}{dr} = -\frac{g m_H \mu}{kT}$$

and: $H = \frac{kT}{g m_H \mu}$

pressure scale height

H(Sun) = 150 km

solution:

$$\rho(r) = \rho(r_0) e^{-\frac{r - r_0}{H}}$$

$$\frac{dP}{dr} = -\rho(r)g(r)$$

$$\rho(r) = \rho(r_0)\, e^{-\frac{r-r_0}{H}}$$

Approximate solution:
Barometric formula

- **exponential decline of density**
- **scale length H ~ T/g** $H = \frac{kT}{g m_H \mu}$
- **explains why atmospheres are so thin**

b. radial flow

Let us now consider the case in which there are deviations from hydrostatic equilibrium and v(r,t) ≠ 0.

Newton's law:

$$dm \frac{d}{dt} v(r,t) = \sum_i df_i$$

$$\frac{d}{dt} v(r,t) = \lim_{\Delta t \to 0} \frac{v(r+\Delta r, t+\Delta t) - v(r,t)}{\Delta t}$$

Taylor expansion →

$$v(r+\Delta r, t+\Delta t) = v(r,t) + \frac{\partial v}{\partial r}\Delta r + \frac{\partial v}{\partial t}\Delta t$$

and

$$\Delta r = v \, \Delta t$$

$$\frac{d}{dt} v(r,t) = \lim_{\Delta t \to 0} \frac{v \frac{\partial v}{\partial r}\Delta t + \frac{\partial v}{\partial t}\Delta t}{\Delta t} = \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r}$$

In general:

$$\frac{d}{dt}\alpha(r,t) = \frac{\partial}{\partial t}\alpha(r,t) + v(r,t) \times grad\{\alpha(r,t)\}$$

$$\frac{d}{dt}v(r,t) = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r}$$

For the assumption of stationarity $\partial/\partial t = 0$

$$\frac{d}{dt}v(r,t) = v\frac{\partial v}{\partial r}$$

$$dm\frac{d}{dt}v(r,t) = dm\,v\frac{\partial v}{\partial r} = \sum_i df_i$$

$$dm = A\rho\,dr \qquad\qquad -g(r)dm - A\frac{dP}{dr}dr + g_{rad}dm$$

$$\frac{dP}{dr} = -\rho[g(r) - g_{rad} + v\frac{dv}{dr}]$$

new term

equation of continuity, mass conservation

$$\dot{M} = \frac{d}{dt}M = 4\pi r^2 \rho \upsilon$$

$$\upsilon = \frac{\dot{M}}{4\pi r^2 \rho}$$

$$\frac{d\upsilon}{dr} = -\frac{2\dot{M}}{4\pi r^3 \rho} - \frac{\dot{M}}{4\pi r^2 \rho^2}\frac{d\rho}{dr} = -\frac{2\upsilon}{r} - \frac{\upsilon}{\rho}\frac{d\rho}{dr}$$

From: $$\frac{dP}{dr} = -\rho[g(r) - g_{rad} + \upsilon\frac{d\upsilon}{dr}]$$

$$\rho\upsilon\frac{d\upsilon}{dr} = -2\rho\frac{\upsilon^2}{r} - \upsilon^2\frac{d\rho}{dr}$$

and from the equation of state:

$$\frac{dP}{dr} = \frac{kT}{m_H \mu}\frac{d\rho}{dr} = \upsilon_{sound}^2\frac{d\rho}{dr}$$

Assuming again that the temperature gradient is small

$$(v_{sound}^2 - v^2)\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\rho[g(r) - g_{rad} - 2\frac{v^2}{r}]$$

Hydrodynamic equation of motion

with

$$escape\ velocity: \frac{m v_{esc}^2(r)}{2} \Rightarrow G\frac{mM}{r} = mrg \rightarrow v_{esc}^2(r) = 2gr$$

re-write

$$(v_{sound}^2 - v^2)\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\rho g(r)[1 - \frac{g_{rad}}{g(r)} - 4\frac{v^2}{v_{esc}^2(r)}]$$

When $v \ll v_{sound}$: practically hydrostatic solution ($v = 0$) for density stratification. This is reached well below the sonic point (where $v = v_{sound}$).

example: v_{sound} = 6 km/s for solar photosphere , 20 km/s for O stars

v_{esc} = 100 to 1000 km/s for main sequence and supergiant stars

→ $(v_{sound}/v_{esc})^2 \ll 1$

3. Conservation of energy

Stellar interior: production of energy via nuclear reactions

Stellar atmosphere: negligible production of energy

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the energy flux is conserved at any given radius

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$$F(r) = {}^{energy}\!/_{area \cdot time}$$

$$4\pi r^2 F(r) = const = luminosity \;\; L$$

In spherical symmetry: $r^2 F(r) = const$

$F(r) \sim 1/r^2$

Plane-parallel: $r^2 \sim R^2 \sim const$

$F(r) \sim const$

4. Concepts of Thermodynamics

Radiation field

Consider a closed“cavity” in thermodynamic equilibrium TE
(photons and particles in equilibrium at some temperature T). →

The specific intensity emitted (through a small hole) is
(energy per area, per unit time, per unit frequency, per unit solid angle)
(**Planck function**):

$$I_\nu \, d\nu = B_\nu(T) \, d\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} d\nu \qquad (\mathrm{erg\,cm^{-2}\,s^{-1}Hz^{-1}sr^{-1}})$$

black body radiation: universal function dependent only on T
not on chemical composition, direction, place, etc.

Note: stars don’t radiate as blackbodies, since they are not closed systems !!!
But their radiation follows the Planck function qualitatively

properties of Planck function

a. $B_\nu(T_1) > B_\nu(T_2)$ (monotonic) for every $T_1 > T_2$ (no function crossing)

b. ν_{max}/T = const Wien's displacement law

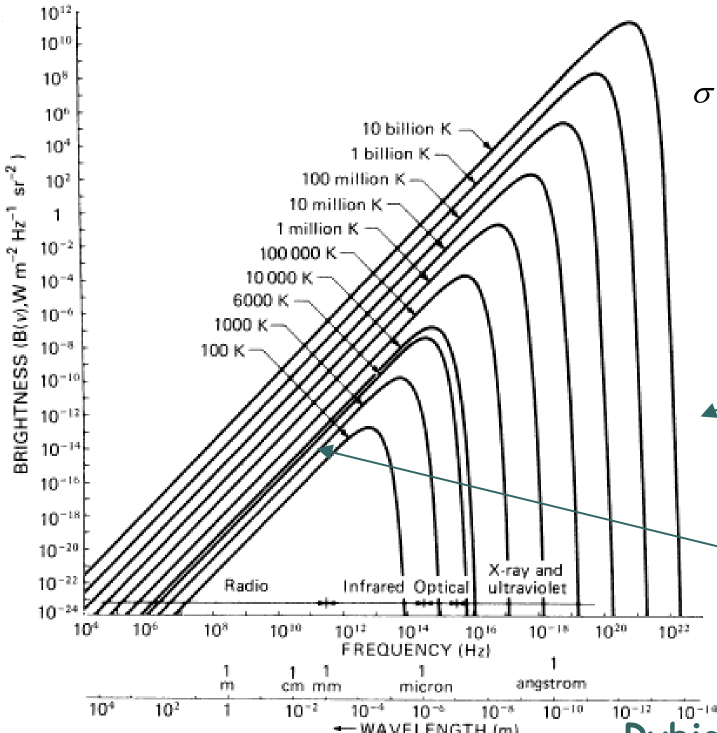
$\lambda_{max} = \frac{2.898 \cdot 10^7}{T} \text{Å}$ for B_λ

$\lambda_{max} = \frac{5.099 \cdot 10^7}{T} \text{Å}$ for B_ν

c. Stefan-Boltzmann law:
(total flux)

$$F = \pi \int_0^\infty B_\nu(T)\, d\nu = \sigma T^4$$

$$\sigma = 5.67 \cdot 10^{-5} \quad erg\, cm^{-2}\, s^{-1}\, K^{-4}$$



Rybicki & Lightman, 79

hν/kT >> 1: Wien

$$B_\nu(T) \approx \frac{2h\nu^3}{c^2} e^{-h\nu/kT}$$

hν/kT << 1: Rayleigh-Jeans

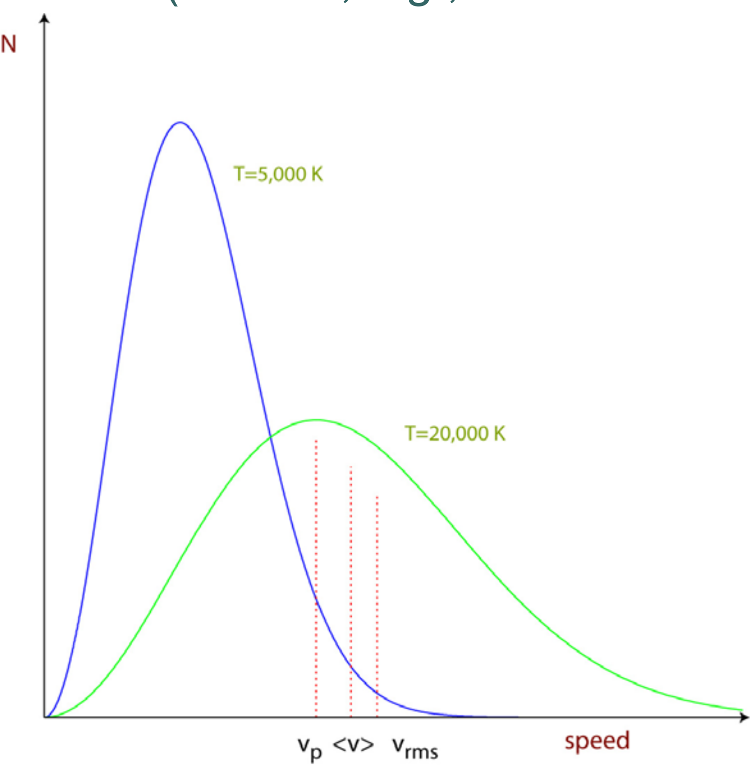
$$B_\nu(T) \approx \frac{2\cancel{h}\nu^2}{c^2} kT$$

Concepts of Thermodynamics

Gas particles

1. Velocity distribution

In complete equilibrium this is given by Maxwell distribution (needed, e.g., for collisional rates):



$$f(v)dv = 4\pi v^2 \left(\frac{m}{2\pi kT} \right)^{3/2} e^{-\frac{mv^2}{2kT}} dv$$

most probable speed: $v_p = \left(2kT/m\right)^{1/2}$

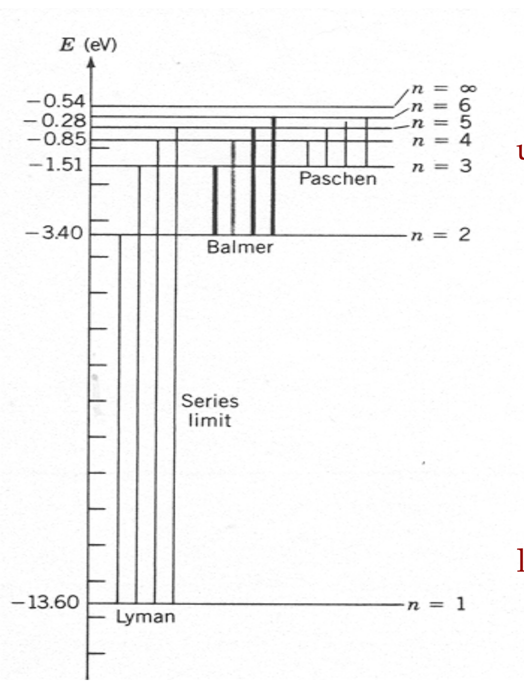
average speed: $\langle v \rangle = \left(8kT/\pi m\right)^{1/2}$

r.m.s. speed: $v_{rms} = \left(3kT/m\right)^{1/2}$

2. Energy level distribution

Boltzmann's equation for the population density of excited states in TE

$$\frac{n_u}{n_l} = \frac{g_u}{g_l} e^{-\frac{(E_u - E_l)}{kT}}$$



upper level: *u*

lower level: *l*

g_u, g_l: statistical weights = 2J+1: number of degenerate states)

= $2n^2$ for Hydrogen

$$\frac{n_n}{n_{tot}} = \frac{g_n}{u} e^{-\frac{E_n}{kT}}$$

$$u = \sum_i g_i e^{-\frac{E_i}{kT}}$$ partition function

3. Kirchhoff’s law

For a thermal emitter in TE

$$\frac{\varepsilon_\nu}{\kappa_\nu} = \frac{\text{emission coefficient}}{\text{absorption coefficient}} = B_\nu(T)$$

as a consequence of Boltzmann’s equation for excitation and Maxwell’s velocity law.

Local Thermodynamic Equilibrium

A star as a whole (or a stellar atmosphere) is far from being in thermodynamic equilibrium: energy is transported from the center to the surface, driven by a temperature gradient.

But for sufficiently small volume elements dV, we can assume TE to hold at a certain temperature $T(r)$

↓

Local **T**hermodynamic **E**quilibrium (LTE)

good approx: stellar interior (high density, small distance travelled by photons, nearly-isotropic, thermalized radiation field)

bad approx: gaseous nebulae (low density, non-local radiation field, optically thin)

stellar atmospheres: optically thick but moderate density

Local Thermodynamic Equilibrium

1. Energy distribution of the gas

determined by **local temperature T(r)**
appearing in Maxwell's/ Boltzmann's equations, and Kirchhoff's law

2. Energy distribution of the photons

photons are carriers of non-local energy through the atmosphere
→ **$I_\nu(r) \neq B_\nu[T(r)]$**

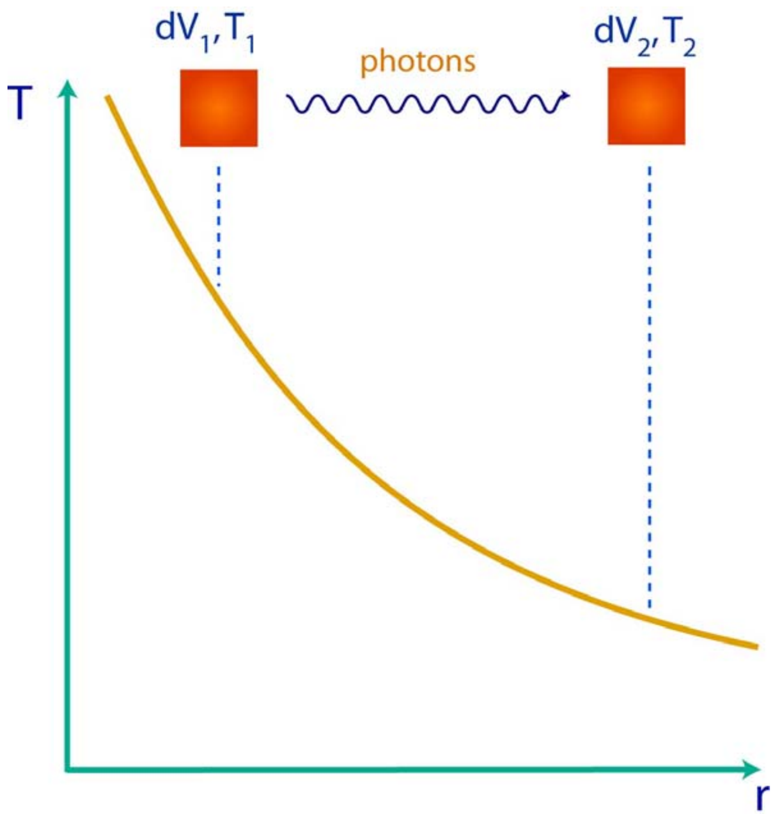
I_ν is a superposition of Planck functions originating at different depths in the atmosphere → radiation transfer

LTE vs NLTE

LTE

each volume element separately in thermodynamic equilibrium at temperature T(r)

1. f(v) dv = Maxwellian with T = T(r)
2. Saha: $(n_p\, n_e)/n_1 \sim T^{3/2} \exp(-h\nu_1/kT)$
3. Boltzmann: $n_i / n_1 = g_i / g_1 \exp(-h\nu_{1i}/kT)$



However:

volume elements not closed systems, interactions by photons

➔ LTE non-valid if absorption of photons disrupts equilibrium

LTE vs NLTE

NLTE if

rate of photon absorptions >> rate of electron collisions

$I_\nu (T) \sim T^\alpha, \ \alpha > 1 \quad \gg \quad n_e \, T^{1/2}$

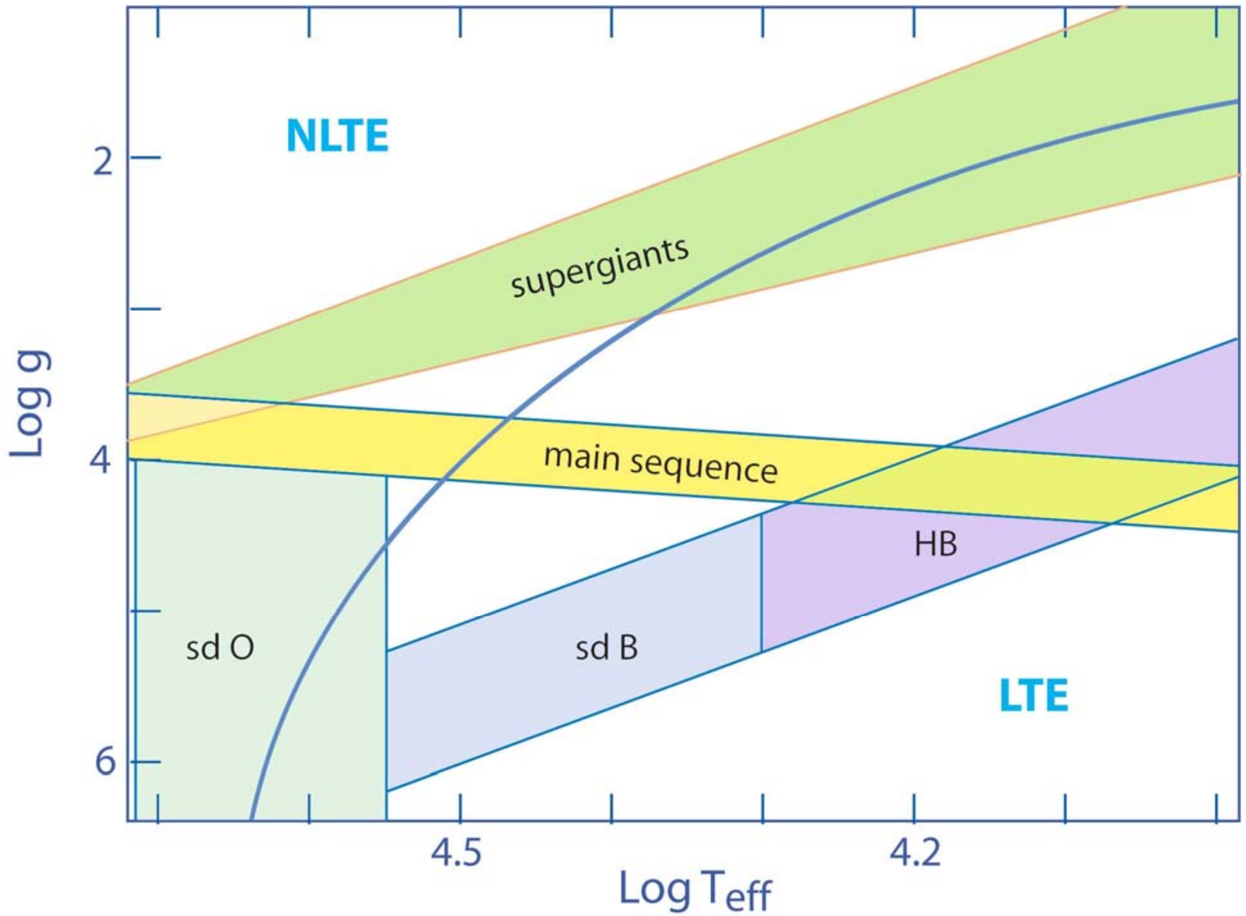
⇓

LTE

valid: low temperatures & high densities

non-valid: high temperatures & low densities

LTE vs NLTE in hot stars



Kudritzki 1978