Some fundamentals

Statistical mechanics

We have seen that the collision timescale for gas in this room is very small relative to radiative timescales such as spontaneous emission. The frequent collisions distribute the energy of the system both in bulk motion (temperature) and internal energy states. The properties of the gas can then be described by the equations of statistical mechanics and thermodynamics. Namely, the speeds of the gas particles follow a Maxwell distribution,

\[ f(v) = \sqrt{\frac{2}{\pi}} \left( \frac{m}{kT} \right)^{3/2} v^2 e^{-mv^2/kT}. \] (1)

Further, the population of energy levels follows a Boltzmann distribution,

\[ \frac{n_u}{n_l} = \frac{g_u}{g_l} e^{-(E_u-E_l)/kT}, \] (2)

where \( g_u \) and \( g_l \) are the statistical weights of the upper and lower levels, and the radiation field follows the Planck function,

\[ B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}. \] (3)

The temperature \( T \) in these equations is the same in all case for local thermodynamic equilibrium (LTE)\(^2\).

The non-equilibrium ISM

The ISM is generally not in thermodynamic equilibrium due to its low density and consequent low collision rates and low optical depths. Collisions between particles may result in radiative decay rather than redistributing internal energies. Photons can escape the system, or enter from external sources. Whereas the distribution of velocities remains generally Maxwellian, and described by a kinetic temperature on scales greater than a mean free path, the distribution

*Similarly, the Saha equation describes the distribution of ionization states but we will not need it in the contexts discussed in this class.
of energy levels may be significantly from the Boltzmann distribution. This can be formulated either as departure coefficients,

$$b_i = \frac{n_{i,\text{actual}}}{n_{i,\text{LTE}}}$$

(4)

or, more commonly, by defining an excitation temperature, $T_{\text{ex}}$, such that

$$\frac{n_u}{n_l} = \frac{g_u}{g_l} e^{-(E_u-E_l)/kT_{\text{ex}}}.$$  

(5)

Note that although $T_{\text{ex}}$ has units of K, it is not a physical temperature and may not be equal to the kinetic temperature. It is a function of the energy level and parameterizes how far the distribution of states if from Boltzmann.

The interstellar radiation field is very dilute but populated by discrete energetic sources (e.g. OB stars) and can contain multiple spectral features from the gas. It is therefore generally very different from the Planck function which we can parameterize in terms of a brightness temperature (see the following subsection).

Despite these complications, we can still look for a statistical equilibrium solution to the distribution of energy levels,

$$\frac{dn_j}{dt} = \sum_i (R_{ij}n_i - R_{ji}n_j) = 0,$$

(6)

where $R_{ij}$ is the (radiative plus collisional) rate from level $i$ to $j$. This matching of forward and reverse rates is also known as the principle of detailed balance. To proceed, however, requires that we understand how radiation interacts with matter.

Radiative transfer

As radiation passes through space, it can be absorbed or scattered by the gas and dust and, conversely, the ISM can emit radiation. Telescopes measure the amount of energy, $dE$, in a particular frequency range, $[v, v+dv]$ that crosses a detector area, $dA$, in time $dt$ from a finite solid angle on the sky, $d\Omega$. These are also the same parameters that we need to consider as we consider the interaction of radiation with matter. We therefore define the specific intensity, $I_\nu$, such that

$$dE = I_\nu dA \cos \theta d\Omega dv dt,$$

(7)

where $\theta$ is the angle between the direction of the radiation and the normal vector to the area. The cgs units of $I_\nu$ are a bit of a mouthful, erg cm$^{-2}$ s$^{-1}$ Hz$^{-1}$ steradian$^{-1}$, but we will replace this by talking about flux measurements in Janskys later on.

If we ignore scattering for now and consider the passage of radiation through a slab of matter, the specific intensity will decrease by

![Figure 2](image.png)

Figure 2: Copy of Draine Figure 7.1 showing the geometry for the equation of radiative transfer.
a fractional amount due to absorption and increase by an absolute amount due to emission by the material. We parameterize this as,

\[ dI_n = -\kappa_n I_n ds + j_n ds, \quad (8) \]

where \( \kappa_n \) is the absorption coefficient per unit length and \( j_n \) is the emissivity per unit length.

Radiation is not affected by how far it travels, only by the interaction with matter. This motivates the change from pathlength to optical depth,

\[ d\tau_n \equiv \kappa_n ds. \quad (9) \]

This now leads to the equation of radiative transfer,

\[ \frac{dI_n}{d\tau_n} = -I_n + S_n, \quad (10) \]

where

\[ S_n \equiv \frac{j_n}{\kappa_n} \quad (11) \]

is the source function. In a purely mathematical sense, this can be readily solved,

\[ I_n(\tau_n) = I_n(0)e^{-\kappa_n} + \int_0^{\tau_n} S_n e^{(\tau_n - \tau')} d\tau'. \quad (12) \]

For matter in thermodynamic equilibrium at temperature \( T \), the specific intensity, \( I_n = B_n(T) \), and \( dI_n = 0 \). Therefore the source function, \( S_n = B_n \) (or equivalently the emissivity \( j_n \) equals the absorptivity, \( \kappa_n B_n \)). This is Kirchoff's Law and, because \( j_n, \kappa_n \), and therefore \( S_n \) are fundamental properties of the matter, this condition must hold in general, not just in equilibrium, but where we replace \( T \) with the excitation temperature. Further, if the excitation conditions are constant along the line of sight, we can integrate the above to

\[ I_n = I_n(0)e^{-\tau_n} + B_n(T_{ex})(1 - e^{-\tau_n}). \quad (13) \]

This is certainly an approximation in many cases but it has utility and will appear in various guises throughout the course.

The flux is the total intensity coming from solid angle \( d\Omega \),

\[ F_n = \int_{\Omega} I_n d\Omega, \quad (14) \]

and we often express this in the unit of Jansky (named after one of the pioneers of radio astronomy) where \( 1 \text{ Jy} = 10^{-23} \text{ erg s}^{-1} \text{ cm}^2 \text{ Hz}^{-1} \).

The monochromatic luminosity is

\[ L_n = 4\pi d^2 F_n \quad (15) \]
where \( d \) is the distance to the source, and the bolometric luminosity is

\[
L_{\text{bol}} = \int_{\nu} L_{\nu} \, d\nu.
\]  

(16)

With the definition of specific intensity and noting the Rayleigh-Jeans limit, \( B_{\nu}(T) \to 2kT/\lambda^2 \) for \( h\nu \ll kT \), we can now also define the afore-mentioned brightness temperature,

\[
T_B = \frac{2k}{\lambda^2} I_{\nu}.
\]

(17)

We can also talk about the specific intensity in terms of wavelength instead of frequency. The relation between them is

\[
I_{\nu} = I_{\lambda} \left| \frac{d\lambda}{d\nu} \right|.
\]

(18)

This then implies

\[
\nu F_{\nu} = \lambda F_{\lambda}
\]

(19)

and

\[
\nu L_{\nu} = \lambda L_{\lambda}
\]

(20)

which are oft-plotted quantities when showing the spectral energy distribution (SED) of an astronomical object.

**Fluid dynamics**

Although the collision rates are low compared to many spontaneous emission rates in the ISM, the mean free path,

\[
l = 1/n\sigma \sim 1/[(1 \text{ cm}^{-3})\pi(1 \text{Angstrom})^2] \sim 3 \times 10^{15} \text{ cm} \simeq 100 \text{ AU},
\]

(21)

is still much smaller than typical size scales of interest (e.g., gravitational collapse of dense molecular cores, expansion of HII regions and supernovae). Consequently we can define macroscopic quantities such as density (\( \rho \)), velocity (\( \mathbf{v} \)), temperature (\( T \)) and pressure (\( P \)) and use fluid equations to describe ISM dynamics. The (non-magnetic) fluid equations come from consideration of the conservation of mass and momentum:

\[
\frac{\partial \rho}{\partial t} + \nabla. (\rho \mathbf{v}) = 0,
\]

(22)

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\nabla P}{\rho} - \nabla \phi.
\]

(23)

The gravitational potential, \( \phi \), is related to the density via Poisson’s equation,

\[
\nabla^2 \phi = 4\pi G \rho.
\]

(24)
These are also known as the Navier-Stokes equations and a full derivation can be found in many textbooks\(^3\).

A third equation describes the energy of the system. Depending on how quickly the gas radiatively cools relative to dynamical changes in the macroscopic quantities, the fluid motions can be considered either isothermal or adiabatic. The former means that radiative cooling keeps the gas at a constant temperature. In this case \( PV = NkT \) =constant or \( P \propto \rho \). Adiabatic conditions are the opposite extreme where the cooling time is much longer than dynamical timescales and the energy can be considered to be conserved. Changes in the pressure and volume are related to the internal energy, \( U \), via the first law of thermodynamics, \( dU = PdV \). For a gas with \( n_f \) degrees of freedom,

\[
U = \frac{n_f}{2} NkT = \frac{n_f}{2} PV,
\]

\[
\implies dU = \frac{n_f}{2} (PdV + VdP) = -PdV
\]

\[
\implies \frac{dP}{P} = -\gamma \frac{dV}{V}, \quad \text{where } \gamma = 1 + 2/n_f
\]

\[
\implies P \propto V^{-\gamma} \propto \rho^\gamma.
\]

The relation between pressure and density, \( P \propto \rho^\gamma \), is known as the equation of state. For monoatomic gas, \( n_f = 3 \) and \( \gamma = 5/3 \). Diatomic molecules have an extra two degrees of freedom, \( n_f = 5 \) so \( \gamma = 7/5 \). The isothermal case can be considered an extension of this description where the \( n_f \rightarrow \infty \) and \( \gamma \rightarrow 1 \), i.e. the gas has so many degrees of freedom that it radiates very efficiently and the gas maintains a constant temperature.

We now solve the 1-d fluid equations for a given equation of state assuming small perturbations on a static solution,

\[
v = v_0 + v_1, \quad \rho = \rho_0 + \rho_1, \quad P = P_0 + P_1.
\]

Choosing a frame of reference such that \( v_0 = 0 \) and dropping second order terms,

\[
\frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial v_1}{\partial x} = 0,
\]

\[
\frac{\partial v_1}{\partial t} + \frac{1}{\rho_0} \frac{\partial P_1}{\partial x} = 0.
\]

The small pressure change, \( P_1 \), induces a density change, \( \rho_1 \) (or vice versa), related by

\[
P_1 = \Delta P = \gamma P \Delta \rho / \rho \leq (\gamma \rho_0 / \rho_0) \rho_1 \equiv a^2 \rho_1,
\]

where \( a^2 = \gamma \rho_0 / \rho_0 = \gamma kT/m \) is the adiabatic sound speed. Now differentiating the first equation with respect to \( t \) and the second with
respect to $x$ and subtracting gives

$$\frac{\partial^2 \rho_1}{\partial t^2} - a^2 \frac{\partial^2 \rho_1}{\partial x^2} = 0.$$ \hspace{1cm} (30)

This has the general solution, $\rho_1 = f(x - at) + g(x + at)$, where $f$ and $g$ are any (twice differentiable) functions\(^4\). What this means is that any small perturbation to the fluid equations at $(x_1, t_1)$ looks the same at $(x_2, t_2)$ where $x_1 \pm t_1 = x_2 \pm t_2$. That is, the functional shape travels along at a speed $x/t = \pm a$. That is why we call $a$ the sound speed and why the above is known as the \textbf{wave equation}.\(^4\)Try it and see!